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Dynamics of Dilute Gases at Equilibrium: From the Atomistic Description to Fluctuating Hydrodynamics

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Abstract. We derive linear fluctuating hydrodynamics as the low density limit of a deterministic system of particles at equilibrium. The proof builds upon the results of Bodineau et al. (Long-time correlations for a hard-sphere gas at equilibrium, 2022) where the asymptotics of the covariance of the fluctuation field is obtained, and on the proof of the Wick rule for the fluctuation field in Bodineau et al. (Long-time derivation at equilibrium of the fluctuating Boltzmann equation, 2022).

1. The Different Levels of Modeling

1.1. The Atomistic Description

The microscopic model consists of identical hard spheres of unit mass and of diameter ε. The motion of N such hard spheres is ruled by a system of ordinary differential equations, which are set in $(\mathbb{T}^d \times \mathbb{R}^d)^N$ where $\mathbb{T}^d$ is the unit d-dimensional periodic box with $d \geq 3$: writing $\mathbf{x}_i^\varepsilon \in \mathbb{T}^d$ for the position of the center of the particle labeled by i and $\mathbf{v}_i^\varepsilon \in \mathbb{R}^d$ for its velocity, one has

$$\frac{\mathrm{d}\mathbf{x}_i^\varepsilon}{\mathrm{d}t} = \mathbf{v}_i^\varepsilon\,, \quad \frac{\mathrm{d}\mathbf{v}_i^\varepsilon}{\mathrm{d}t} = 0 \quad \text{as long as } |\mathbf{x}_i^\varepsilon(t) - \mathbf{x}_j^\varepsilon(t)| > \varepsilon \quad \text{for } 1 \leq i \neq j \leq N\,, \tag{1.1}$$

This paper is dedicated to the memory of K. Gawedzki. We show how dynamical noise in hydrodynamic models of perfect gases can emerge from a deterministic microscopic dynamics, by combination of strong microscopic instabilities with the small-scale regularity of the initial distribution. This enhanced dynamical noise seems to share similarities with the concept of spontaneous stochasticity introduced by K. Gawedzki et al. in [4].

with specular reflection at collisions:

$$\left.\begin{aligned} (\mathbf{v}_i^\varepsilon)' &:= \mathbf{v}_i^\varepsilon - \frac{1}{\varepsilon^2}(\mathbf{v}_i^\varepsilon - \mathbf{v}_j^\varepsilon)\cdot(\mathbf{x}_i^\varepsilon - \mathbf{x}_j^\varepsilon)\,(\mathbf{x}_i^\varepsilon - \mathbf{x}_j^\varepsilon) \\ (\mathbf{v}_j^\varepsilon)' &:= \mathbf{v}_j^\varepsilon + \frac{1}{\varepsilon^2}(\mathbf{v}_i^\varepsilon - \mathbf{v}_j^\varepsilon)\cdot(\mathbf{x}_i^\varepsilon - \mathbf{x}_j^\varepsilon)\,(\mathbf{x}_i^\varepsilon - \mathbf{x}_j^\varepsilon) \end{aligned}\right\} \quad \text{if } |\mathbf{x}_i^\varepsilon(t) - \mathbf{x}_j^\varepsilon(t)| = \varepsilon\,. \tag{1.2}$$

This flow does not cover all possible situations, as multiple simultaneous collisions are excluded. But it can be shown (see [1]) that for almost every admissible initial configuration $(\mathbf{x}_i^{\varepsilon 0}, \mathbf{v}_i^{\varepsilon 0})_{1\le i\le N}$, there are neither multiple simultaneous collisions, nor accumulations of collision times, so that the dynamics is globally well defined.

We will not be interested here in one specific realization of this deterministic dynamics, but rather in a statistical description. This is achieved by introducing a measure at time 0, on the phase space we now specify. The collections of N positions and velocities are denoted respectively by $X_N := (x_1, \dots, x_N)$ in $\mathbb{T}^{dN}$ and $V_N := (v_1, \dots, v_N)$ in $\mathbb{R}^{dN}$, and we set $Z_N := (X_N, V_N)$, with $Z_N = (z_1, \dots, z_N)$, $z_i = (x_i, v_i)$. A set of N particles is characterized by a random variable $\mathbf{Z}_N^{\varepsilon 0} = (\mathbf{z}_1^{\varepsilon 0}, \dots, \mathbf{z}_N^{\varepsilon 0})$ specifying the time-zero configuration in the phase space

$$\mathcal{D}_N^\varepsilon := \left\{ Z_N \in (\mathbb{T}^d \times \mathbb{R}^d)^N \,/\, \forall i \neq j\,, \quad |x_i - x_j| > \varepsilon \right\}, \tag{1.3}$$

and an evolution

$$t \longmapsto \mathbf{Z}_N^\varepsilon(t) = \left(\mathbf{z}_1^\varepsilon(t), \dots, \mathbf{z}_N^\varepsilon(t)\right)\,, \qquad t > 0$$

according to the deterministic flow (1.1)–(1.2) (well defined with probability 1).

To avoid spurious correlations due to a given total number of particles, we actually consider a grand canonical state (as in [21,28]), set on the phase space

$$\mathcal{D}^\varepsilon := \bigcup_{N\ge 0} \mathcal{D}_N^\varepsilon$$

(notice that $\mathcal{D}_N^\varepsilon = \emptyset$ for N large). This means that the total number of particles is also a random variable, which we shall denote by $\mathcal{N}$.

More precisely, at equilibrium the probability density of finding N particles at configuration Z_N is given by

$$\frac{1}{N!} W_N^\varepsilon(Z_N) := \frac{1}{\mathcal{Z}^\varepsilon}\frac{\mu_\varepsilon^N}{N!}\,\mathbf{1}_{\mathcal{D}_N^\varepsilon}(Z_N)\,\mathcal{M}^{\otimes N}(V_N)\,, \qquad \text{for } N = 0, 1, 2, \dots \tag{1.4}$$

for some (large) μ_ε to be fixed below, with

$$\mathcal{M}(v) := \frac{1}{(2\pi)^{\frac{d}{2}}}\exp\left(-\frac{|v|^2}{2}\right)\,, \qquad \mathcal{M}^{\otimes N}(V_N) = \prod_{i=1}^N \mathcal{M}(v_i)\,,$$

and the partition function is given by

$$\mathcal{Z}^\varepsilon := 1 + \sum_{N\ge 1}\frac{\mu_\varepsilon^N}{N!}\int_{\mathbb{T}^{dN}} \prod_{i\neq j} \mathbf{1}_{|x_i - x_j| > \varepsilon}\, \mathrm{d}X_N\,.$$

Here and below, $\mathbf{1}_A$ will be the characteristic function of the set A. The probability of an event A with respect to the equilibrium measure (1.4) will be denoted $\mathbb{P}_\varepsilon(A)$, and $\mathbb{E}_\varepsilon$ will be the expected value. Definition (1.4) ensures that

$$\mu_\varepsilon^{-1}\mathbb{E}_\varepsilon\left(\mathcal{N}\right) \to 1$$

as $\mu_\varepsilon \to \infty$ with $\mu_\varepsilon \varepsilon^d \ll 1$.

1.2. The Kinetic Description

Let us define the empirical measure of the hard-sphere model

$$\pi_t^\varepsilon := \frac{1}{\mu_\varepsilon} \sum_{i=1}^{\mathcal{N}} \delta_{\mathbf{z}_i^\varepsilon(t)} \,. \tag{1.5}$$

Under the invariant measure (1.4), it is not hard to see that if $\mu_\varepsilon \varepsilon^d \to 0$ then π_t^ε concentrates on $\mathcal{M}$: for any test function $h : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ and any $\delta > 0$, $t \in \mathbb{R}$,

$$\mathbb{P}_\varepsilon\left(\left|\pi_t^\varepsilon(h) - \int_{\mathbb{T}^d\times\mathbb{R}^d} \mathcal{M}(v)h(z)\right| > \delta\right) \xrightarrow[\mu_\varepsilon\to\infty]{} 0\,, \tag{1.6}$$

which can be interpreted as a law of large numbers.

The fluctuations of the empirical density π_t^ε around its equilibrium value are described by the fluctuation field ζ_t^ε defined by

$$\zeta_t^\varepsilon(h) := \sqrt{\mu_\varepsilon}\left(\pi_t^\varepsilon(h) - \mathbb{E}_\varepsilon\big(\pi_t^\varepsilon(h)\big)\right)\,, \tag{1.7}$$

for any test function h. Initially ζ_0^ε converges in law towards a Gaussian white noise ζ_0 with covariance

$$\mathbb{E}\big(\zeta_0(h_1)\,\zeta_0(h_2)\big) = \int h_1(z)\,h_2(z)\,\mathcal{M}(v)\,\mathrm{d}z\,. \tag{1.8}$$

As the measure is invariant, this covariance is constant in time. Let us define the mean free path

$$\alpha := (\mu_\varepsilon \varepsilon^{d-1})^{-1}\,,$$

and assume that $\alpha^{-1} \geq 1$ is bounded or slowly diverging, corresponding to the low density scaling. In this scaling it has been proved in [11,12] that $(\zeta_t^\varepsilon)_{[0,T]}$ converges in law for all times T to a weak solution of the fluctuating Boltzmann equation

$$\mathrm{d}\zeta_t = \left(-v\cdot\nabla_x - \frac{1}{\alpha}\mathcal{L}\right)\zeta_t\,\mathrm{d}t + \mathrm{d}\eta_t\,, \tag{1.9}$$

where the linearized collision operator is given by

$$\mathcal{L}g(v) := \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \mathcal{M}(w)\big((v-v_*)\cdot\omega\big)_+ \left[g(v)+g(v_*)-g(v')-g(v'_*)\right]\mathrm{d}v_*\,\mathrm{d}\omega \tag{1.10}$$

with notation

$$v' = v - ((v-v_*)\cdot\omega)\,\omega\,,\quad v'_* = v_* + ((v-v_*)\cdot\omega)\,\omega \tag{1.11}$$

for the precollisional velocities obtained upon scattering, and $d\eta_t(x,v)$ is a stationary Gaussian noise, explicitly characterized (see [25]). It has zero mean and covariance

$$\begin{aligned}&\mathbb{E}\left(\int_0^T \mathrm{d}t \int \mathrm{d}z h_1(z)\eta_t(z)\int_0^T \mathrm{d}t_* \int \mathrm{d}z_*\, h_2(z_*)\eta_{t_*}(z_*)\right)\\&\quad = \frac{1}{2\alpha}\int_0^T \mathrm{d}t \int \mathrm{d}\mu(z,z_*,\omega)\mathcal{M}(v)\,\mathcal{M}(v_*)\Delta h_1\,\Delta h_2\end{aligned} \tag{1.12}$$

denoting

$$\mathrm{d}\mu(z,z_*,\omega) := \delta_{x-x_*}\,\big((v-v_*)\cdot\omega\big)_+ \mathrm{d}\omega\,\mathrm{d}v\,\mathrm{d}v_*\mathrm{d}x$$

and defining

$$\Delta h_j(z,z_*,\omega) := h_j(z') + h_j(z'_*) - h_j(z) - h_j(z_*)\,,$$

where $z'_i := (x_i, v'_i)$ with notation (1.11) for the velocities obtained upon scattering. Note that this noise is white in time and space, but correlated in velocities.

1.3. The Hydrodynamic Description

It is by now classical (see [2,7,17] and references therein) that the solutions to the scaled linearized Boltzmann equation

$$\partial_t g_\alpha + v\cdot\nabla_x g_\alpha + \frac{1}{\alpha}\mathcal{L}g_\alpha = 0\,, \quad g_\alpha(0) = g^0 \tag{1.13}$$

converge in the fast relaxation limit $\alpha \to 0$ towards the local thermodynamic equilibrium

$$g(t,x,v) = \rho(t,x) + u(t,x)\cdot v + \theta(t,x)\frac{|v|^2-d}{2}$$

where ρ, u, θ satisfy the acoustic equations

$$\begin{cases}\partial_t\rho + \nabla_x\cdot u = 0\\ \partial_t u + \nabla_x(\rho+\theta) = 0\\ \partial_t\theta + \frac{2}{d}\nabla_x\cdot u = 0\end{cases} \tag{1.14}$$

and the initial data is the projection of g^0 onto hydrodynamic modes

$$\begin{aligned}&\rho_{|t=0}(x) := \int g_0(x,v)\mathcal{M}(v)\,\mathrm{d}v\,, \quad u_{|t=0}(x) := \int g_0(x,v)v\mathcal{M}(v)\,\mathrm{d}v\,,\\&\theta_{|t=0}(x) := \int g_0(x,v)\left(\frac{|v|^2}{d}-1\right)\mathcal{M}(v)\,\mathrm{d}v\,.\end{aligned}$$

In the linearized equation (1.13), the frequency of collisions $1/\alpha$ has been tuned according to the hyperbolic scaling. The diffusive regime can then be found by rescaling time by a factor $1/\alpha$. In this way, one can also obtain the weak convergence (which actually filters out the fast oscillating acoustic waves)

$$g_\alpha\left(\frac{\tau}{\alpha},x,v\right) \rightharpoonup u(\tau,x)\cdot v + \theta(\tau,x)\frac{|v|^2-(d+2)}{2} \tag{1.15}$$

towards diffusive fluid models, namely the incompressible Stokes-Fourier equations

$$\begin{cases} \partial_\tau u = \nu \Delta_x u, & \nabla_x \cdot u = 0 \\ \partial_\tau \theta = \kappa \Delta_x \theta \,, \end{cases} \tag{1.16}$$

where the diffusion coefficients ν and κ depend only on the linearized collision operator $\mathcal{L}$ (they are defined explicitly in (3.20) below). The initial data is the projection of g^0 onto non-oscillating hydrodynamic modes

$$u_{|\tau=0}(x) := P \int g_0 v \mathcal{M}(v)\, \mathrm{d}v \,, \quad \theta_{|\tau=0}(x) := \int g_0 \left(\frac{|v|^2}{d+2} - 1 \right) \mathcal{M}(v)\, \mathrm{d}v \tag{1.17}$$

where P is the Leray projection on divergence free vector fields. In the following, we refer to non-oscillating modes as those satisfying the incompressibility and Boussinesq constraints (see 3.16).

1.4. Fluctuating Hydrodynamics

In the hyperbolic regime corresponding to (1.14), the fluctuation-dissipation principle predicts that there will be no dynamical fluctuation and the fluctuation field tested against hydrodynamical modes (ρ, u, θ) is simply transported by the acoustic equation. In contrast, in the diffusive regime, when taking into account the noise at kinetic level (i.e. starting with (1.9)), we expect to obtain fluctuating hydrodynamics. In the following, we will focus on this more interesting case. We refer to [26], Section 7.1 for the general theory of hydrodynamic fluctuations, which was first developed for equilibrium states in [23]. The link with the predictions from kinetic theory in the case of dilute gases was discussed in [22] (see also [3] for a recent contribution).

Let us define a joint process by time rescaling and projecting on non-oscillating hydrodynamic modes the fluctuation field ζ^ε_t defined in (1.7). According to (1.15) we consider, for any pair of test functions $(\varphi, \psi) \in C^\infty(\mathbb{T}^d; \mathbb{R}^d \times \mathbb{R})$ with $\nabla_x \cdot \varphi = 0$, the fluctuation field

$$\zeta^\varepsilon_t (\varphi \cdot v) + \zeta^\varepsilon_t \left(\psi \left(\frac{|v|^2}{d+2} - 1 \right) \right) .$$

To simplify the notation, we denote from now on the couple of test functions by

$$\phi = (\varphi, \psi) \in C^\infty(\mathbb{T}^d; \mathbb{R}^d \times \mathbb{R}) \,, \quad \nabla_x \cdot \varphi = 0 \tag{1.18}$$

and to recover a diffusive regime, time is rescaled as follows:

$$\begin{aligned} \xi^\varepsilon_\tau(\phi) &:= \mathcal{U}^\varepsilon_\tau(\varphi) + \Theta^\varepsilon_\tau(\psi) \\ &:= \zeta^\varepsilon_{\tau/\alpha} (\varphi \cdot v) + \zeta^\varepsilon_{\tau/\alpha} \left(\psi \left(\frac{|v|^2}{d+2} - 1 \right) \right) . \end{aligned} \tag{1.19}$$

We stress the fact that in contrast with ζ^ε, the test functions in ξ^ε only depend on the space variable. In the limit $\mu_\varepsilon \to \infty$ with α slowly vanishing, we expect the fluctuation fields $\big(\mathcal{U}^\varepsilon, \Theta^\varepsilon\big)$ to converge in the sense of distributions to $(\mathcal{U}, \Theta)$ solving the fluctuating Stokes-Fourier equations

$$\begin{cases} \partial_\tau \mathcal{U} = \nu \Delta_x \mathcal{U} + \sqrt{2\nu}\, P\, \nabla \cdot \dot{\mathbb{W}}_t, \\ \partial_\tau \Theta = \kappa \Delta_x \Theta + \sqrt{\frac{4\kappa}{d+2}}\, \nabla \cdot \dot{W}_t\,, \end{cases} \tag{1.20}$$

where W_t is a space/time white noise taking values in $\mathbb{R}^d$ and $\mathbb{W}_t$ is a $d \times d$ matrix with coefficients given by independent white noises. We recall that P stands for the Leray projection on divergence free vector fields. Note that the noise is tuned so that the field has a covariance compatible with the invariance of (1.8). The equations (1.20) should be understood in a weak sense, namely restricting to any pair of test functions $(\varphi, \psi) \in C^\infty(\mathbb{T}^d; \mathbb{R}^d \times \mathbb{R})$ with $\nabla_x \cdot \varphi = 0$

$$\begin{cases} \mathcal{U}_\tau(\varphi) = \mathcal{U}_0(e^{\nu\tau\Delta_x}\varphi) + \sqrt{2\nu} \displaystyle\int_0^\tau \mathrm{d}\sigma \dot{\mathbb{W}}_\sigma \left(\nabla e^{\nu(\tau-\sigma)\Delta_x}\,\varphi\right) \\ \Theta_\tau(\psi) = \Theta_0\left(e^{\kappa\tau\Delta_x}\psi\right) + \sqrt{\frac{4\kappa}{d+2}} \displaystyle\int_0^\tau \mathrm{d}\sigma \dot{W}_\sigma \left(e^{\kappa(\tau-\sigma)\Delta_x}\nabla\psi\right). \end{cases}$$

We stress that the fluctuations in (1.20) exactly compensate the dissipation according to the fluctuation-dissipation principle. In particular, both Gaussian processes are characterized by their covariances for $\sigma \le \tau$

$$\begin{cases} \mathbb{E}\Big(\mathcal{U}_\sigma(\varphi_1)\mathcal{U}_\tau(\varphi_2)\Big) = \displaystyle\int_{\mathbb{T}^d} \mathrm{d}x\, \varphi_1(x) \cdot e^{\nu(\tau-\sigma)\Delta_x}\varphi_2(x) \\ \mathbb{E}\Big(\Theta_\sigma(\psi_1)\Theta_\tau(\psi_2)\Big) = \dfrac{2}{d+2} \displaystyle\int_{\mathbb{T}^d} \mathrm{d}x\, \psi_1(x)\; e^{\kappa(\tau-\sigma)\Delta_x}\psi_2(x). \end{cases} \tag{1.21}$$

The main result of this paper is that both limits $\mu_\varepsilon \to \infty$ with $\mu_\varepsilon \varepsilon^{d-1} = \alpha^{-1}$, and $\alpha \to 0$ can be combined in order to derive fluctuating hydrodynamics directly from the dynamics of particles, thus solving Hilbert's sixth problem in the particular case of fluctuations of perfect gases at equilibrium.

Theorem 1.1. *Consider a system of hard spheres at equilibrium in a d-dimensional periodic box with $d \ge 3$, with inverse mean free time $\alpha^{-1} := \mu_\varepsilon \varepsilon^{d-1} \le \log\log\log \mu_\varepsilon$. Then, in the diffusive limit $\mu_\varepsilon \to \infty, \alpha \to 0$, the rescaled joint process $(\xi^\varepsilon_\tau)_{\tau \in [0,T]}$ defined in (1.19) converges for any $T > 0$ in law to the solution of the fluctuating Stokes-Fourier equations (1.20).*

Although the microscopic dynamics is completely deterministic, Theorem 1.1 shows that the limiting model has two stochastic contributions:

• the initial fluctuation field keeps track of the uncertainty on the initial data, at scale $O(1)$: this corresponds to nothing else than the Gaussian fluctuations under the invariant measure;

• the dynamical noise driving (1.20) is more subtle as it comes from the sensitivity of the particle system to the details of the initial configuration, at smaller scales.

At variance, for one-dimensional integrable systems, one expects that the dominant contribution is the transport of the initial fluctuations with some additional random shift in the large scale limit, as was pointed out recently in [16] for the the hard rod system (see also [14]). The white noise in (1.9) preserves locally the hydrodynamic modes, however at diffusive time scales, it ultimately induces the local noise on the hydrodynamic projections (1.20). Note that a spontaneous generation of noise also holds for the diffusive limits of a tagged particle to a Brownian motion in an equilibrium hard sphere gas [6,8] (see also [15] in the quantum case).

As stated at the beginning of this article, the enhanced dynamical noise appearing in this model seems to share similarities with the concept of spontaneous stochasticity introduced in [4], and it would be very interesting to devise a common formalism to describe both phenomena.

2. The Fluctuation Field in the Low Density Limit: State of the Art, and Strategy of Proof

The present paper relies on the "weak convergence" approach devised in [11,12] in order to prove the convergence of the fluctuation field to the solution of the fluctuating Boltzmann Eq. (1.9). The proofs of [11,12] are quantitative, and the important parameter is the number of collisions, which is proportional to the observation time and inversely proportional to the mean free time α. Thus, the (diffusive) observation time T/α and the parameter α^{-1} can be chosen slowly diverging with μ_ε, for instance as $O(\log\log\log\mu_\varepsilon)$. This will allow us to reach the diffusive regime described in Sect. 1.4. In the rest of this section, we gather the results of [11,12] we shall be using here. We refer to those papers for proofs—see also [13] for an overview.

For the sake of clarity, we will use the following notations for the different time scales described in the previous section :

$$\begin{aligned}&\text{kinetic scale: } t=\alpha t_{\text{kin}} \text{ with } t_{\text{kin}}=O(1), \quad \text{acoustic scale: } t=O(1),\\ &\text{diffusive scale: } t=\tau/\alpha \text{ with } \tau=O(1).\end{aligned} \tag{2.1}$$

2.1. Convergence of the Covariance for Diffusive Times

In the analysis of the fluctuation field for diffusive times, the first step is to study the asymptotic behaviour of the time-rescaled covariance

$$\mathrm{Cov}_\varepsilon\left(\frac{\tau}{\alpha},g^0,h\right):=\mathbb{E}_\varepsilon\left(\zeta_0^\varepsilon(g^0)\,\zeta_{\tau/\alpha}^\varepsilon(h)\right) \tag{2.2}$$

as $\mu_\varepsilon\to\infty$, $\mu_\varepsilon\varepsilon^{d-1}=\alpha^{-1}$. The following result states that this covariance is well approximated on $\mathbb{R}^+$ by $\displaystyle\int \mathcal{M}g_\alpha(\frac{\tau}{\alpha})h\mathrm{d}x\mathrm{d}v$ where g_α is the solution of the scaled linearized Boltzmann Eq. (1.13) starting from $g^0\in L^2_{\mathcal{M}}$, defined by the norm

$$\|g\|_{L^2_{\mathcal{M}}}:=\left(\int_{\mathbb{T}^d\times\mathbb{R}^d}|g|^2\,\mathcal{M}\mathrm{d}x\mathrm{d}v\right)^{\frac{1}{2}}. \tag{2.3}$$

Theorem 2.1 ([11], Linearized Boltzmann equation). *Consider a system of hard spheres at equilibrium in a d-dimensional periodic box with $d\geq 3$. Let g^0*

and h be two Lipschitz functions on $\mathbb{T}^d\times\mathbb{R}^d$ and let g_α be the unique solution in $L^\infty(\mathbb{R}^+;L^2_{\mathcal{M}})$ to (1.13) associated with the initial data g^0. Then, in the low density regime $\mu_\varepsilon\to\infty$, $\mu_\varepsilon\varepsilon^{d-1}=\alpha^{-1}\le\log\log\log\mu_\varepsilon$, the covariance of the fluctuation field $\left(\zeta^\varepsilon_{\tau/\alpha}\right)_{\tau\ge0}$ defined by (2.2) satisfies the following estimate: for any $T>0$ such that $(T/\alpha^2)^{3/2}\ll(\log\log\mu_\varepsilon)^{\frac14}$,

$$\begin{aligned}\sup_{\tau\in[0,T]}&\left|\mathrm{Cov}_\varepsilon\left(\frac{\tau}{\alpha},g^0,h\right)-\int g_\alpha(\frac{\tau}{\alpha})h\,\mathcal{M}\mathrm{d}x\mathrm{d}v\right|\\&\le C\|h\|_{W^{1,\infty}}\|g^0\|_{W^{1,\infty}}\left(\frac{CT}{\alpha^2}\right)^{3/2}(\log\log\mu_\varepsilon)^{-1/4}.\end{aligned}\tag{2.4}$$

Remark 2.1. In accordance with the diffusive scaling, this estimate depends on T/α^2, which is the ratio between the observation time T/α and the mean free time α.

2.2. Convergence of Higher Order Moments for Diffusive Times

The next step is to prove that the process $\zeta^\varepsilon_{\tau/\alpha}$ is asymptotically Gaussian when $\mu_\varepsilon\to\infty$ and $\mu_\varepsilon\varepsilon^{d-1}=\alpha^{-1}\to\infty$. This boils down to showing that the moments are determined by the covariances according to Wick's rule

$$\begin{aligned}\lim_{\substack{\mu_\varepsilon\to\infty\\ \alpha\to0}}&\Bigg|\mathbb{E}_\varepsilon\left[\zeta^\varepsilon_{\tau_1/\alpha}(h_1)\ \dots\ \zeta^\varepsilon_{\tau_p/\alpha}(h_p)\right]\\&-\sum_{\eta\in\mathfrak{S}_p^{\mathrm{pairs}}}\prod_{\{i,j\}\in\eta}\mathbb{E}_\varepsilon\left[\zeta^\varepsilon_{\tau_i/\alpha}(h_i)\ \zeta^\varepsilon_{\tau_j/\alpha}(h_j)\right]\Bigg|=0,\end{aligned}\tag{2.5}$$

uniformly in $\tau_1,\dots,\tau_p\in[0,T]$, where $\mathfrak{S}_p^{\mathrm{pairs}}$ is the set of partitions of $\{1,\dots,p\}$ made only of pairs. Notice that if p is odd then $\mathfrak{S}_p^{\mathrm{pairs}}$ is empty and the product of the moments is asymptotically 0.

Theorem 2.2 ([12], Wick's rule). *Consider a system of hard spheres at equilibrium in a d-dimensional periodic box with $d\ge3$. Let $(h_i)_{1\le i\le p}$ be a family of p bounded functions on $\mathbb{T}^d\times\mathbb{R}^d$. Then, in the low density regime $\mu_\varepsilon\to\infty$, $\mu_\varepsilon\varepsilon^{d-1}=\alpha^{-1}\le\log\log\log\mu_\varepsilon$, for any $T>0$ such that $(T/\alpha^2)^{\frac{2p-1}{2}}\ll$ (log log $\mu_\varepsilon)^{\frac14}$, the fluctuation field $\left(\zeta^\varepsilon_{\tau/\alpha}\right)_{\tau\ge0}$ defined by (1.7) satisfies*

$$\begin{aligned}&\left|\mathbb{E}_\varepsilon\left[\zeta^\varepsilon_{\tau_1/\alpha}(h_1)\ \dots\ \zeta^\varepsilon_{\tau_p/\alpha}(h_p)\right]-\sum_{\eta\in\mathfrak{S}_p^{\mathrm{pairs}}}\prod_{\{i,j\}\in\eta}\mathbb{E}_\varepsilon\left[\zeta^\varepsilon_{\tau_i/\alpha}(h_i)\ \zeta^\varepsilon_{\tau_j/\alpha}(h_j)\right]\right|\\&\qquad\le\left(\prod_{i=1}^p\|h_i\|_{L^\infty}\right)\left(\frac{CT}{\alpha^2}\right)^{(2p-1)/2}(\log\log\mu_\varepsilon)^{-1/4},\end{aligned}\tag{2.6}$$

uniformly in $\tau_1,...\tau_p\in[0,T]$.

2.3. Tightness in the Kinetic Regime

Finally for processes which depend on a continuous variable (the time variable in our setting), the convergence of time marginals is not enough to characterize the convergence in law: possible oscillations with respect to time need to be under control (see [5, Theorem 13.2 page 139]). For the fluctuation field ζ^ε, this tightness property has been obtained for short kinetic times, but actually since the equilibrium measure is invariant under the dynamics, a union bound provides the tightness on any finite kinetic time, i.e. times of order $O(\alpha)$.

For times much longer than kinetic times, we actually do not expect the process ζ^ε_t to be tight. Since the covariance $\mathrm{Cov}_\varepsilon\left(t, g^0, h\right)$ is close to the solution of the scaled linearized Boltzmann equation (1.13), we expect to see a fast relaxation process with rate $O(\frac{1}{\alpha})$, meaning that only the hydrodynamic part of g_α can be compact for $t = O(1)$. Going to diffusive times $t = \tau/\alpha$, we also expect to have acoustic waves producing fast oscillations, meaning that only the non-oscillating hydrodynamic part of g_α can be compact for $\tau = O(1)$. Nevertheless, after projecting on the non-oscillating modes, we are going to show in Sect. 4 that the process $(\xi^\varepsilon_\tau)_{\tau\geq 0}$ defined by (1.19) is tight on the diffusive scale.

2.4. Strategy of the Proof of Theorem 1.1

In view of deriving fluctuating hydrodynamic equations and proving Theorem 1.1, the strategy is now straightforward: we consider the rescaled fluctuation field ξ^ε_τ projected on hydrodynamic, non-oscillating modes (recall (1.19)), and check that with such test functions and this scaling in time, Gaussianity (Theorem 2.2) and tightness still hold, and that the covariance asymptotically converges to the solution to the Stokes-Fourier equation. Note that the projection (1.19) leads to considering test functions which are unbounded in v and therefore there are some technical issues when applying Theorems 2.1 and 2.2. These are dealt with in Sect. 3.2, thanks to a cut-off in energies introduced in Sect. 3.1. The tightness of the process on the diffusive time-scale is derived in Sect. 4.

3. Finite Time Marginals

In this section, we are going to characterize the limiting law of the process by proving the following result. We set from now on

$$\alpha^{-1} = \log\log\log \mu_\varepsilon\,.$$

Proposition 3.1. *For arbitrary times $\tau_1, \dots, \tau_L$ and test functions $\phi^{(1)} = (\varphi^{(1)}, \psi^{(1)}), \dots$ and $\phi^{(L)} = (\varphi^{(L)}, \psi^{(L)})$ chosen as in (1.18), the time marginals $\left(\xi^\varepsilon_{\tau_\ell}\left(\phi^{(\ell)}\right)\right)_{\ell\leq L}$ converge in law to the limiting process $\left(\mathcal{U}_{\tau_\ell}(\varphi^{(\ell)}), \Theta_{\tau_\ell}(\psi^{(\ell)})\right)_{\ell\leq L}$ as μ_ε tends to infinity.*

3.1. Truncated Hydrodynamic Fields

To prove that the limit is Gaussian, Theorem 2.2 cannot be used directly with the process $(\xi_\tau^\varepsilon)_{\tau\geq 0}$ as the test functions are unbounded in L^∞ due to divergences in the velocities. Thus an intermediate cut-off process needs to be introduced. Let us fix an energy cut-off $R \gg 1$ to be determined (see (3.2) below). Recalling (1.19), we define the modified joint process $\bar\xi_\tau^\varepsilon$ as follows. For any test function ϕ as in (1.18), we set

$$\bar\xi_\tau^\varepsilon(\phi) := \zeta_{\tau/\alpha}^\varepsilon\Big(\chi\Big(\frac{|v|^2}{R}\Big)\varphi\cdot v\Big) + \zeta_{\tau/\alpha}^\varepsilon\left(\chi\left(\frac{|v|^2}{R}\right)\psi\left(\frac{|v|^2}{d+2}-1\right)\right), \tag{3.1}$$

where χ is a smooth cut-off function with compact support

$$\chi_{|[0,1]}\equiv 1, \qquad \chi_{|[2,+\infty[}\equiv 0\,.$$

We choose R depending on ε and converging to ∞ as $\mu_\varepsilon\to\infty$ as follows

$$R=\alpha^{-1}=\log\log\log\mu_\varepsilon\,. \tag{3.2}$$

Note that the test functions

$$\bar h := \left(\varphi\cdot v+\psi\left(\frac{|v|^2}{d+2}-1\right)\right)\chi\left(\frac{|v|^2}{R}\right)$$

are smooth and bounded thanks to the cut-off in v:

$$\|\bar h\|_{W^{1,\infty}_{x,v}} \leq CR^2(\|\varphi\|_{W^{1,\infty}_x}+\|\psi\|_{W^{1,\infty}_x})\,. \tag{3.3}$$

The process $\bar\xi_\tau^\varepsilon$ is a good approximation of ξ_τ^ε when $R\to\infty$.

Lemma 3.2. *Setting $\xi_\tau^{\varepsilon,>} := \xi_\tau^\varepsilon - \bar\xi_\tau^\varepsilon$ then for all $1\leq q<\infty$ and for ε small enough*

$$\mathbb{E}_\varepsilon\left[\left(\xi_\tau^{\varepsilon,>}(\phi)\right)^q\right] \leq C_q\|\phi\|^q_{L^q(\mathbb{T}^d)}e^{-R/4}\,. \tag{3.4}$$

Furthermore, one has also

$$\mathbb{E}_\varepsilon\left[(\xi_\tau^\varepsilon(\phi))^q\right] \leq C_q\|\phi\|^q_{L^q(\mathbb{T}^d)} \quad \text{and} \quad \mathbb{E}_\varepsilon\left[\left(\bar\xi_\tau^\varepsilon(\phi)\right)^q\right] \leq C_q\|\phi\|^q_{L^q(\mathbb{T}^d)}. \tag{3.5}$$

As a consequence, the convergence in law of $\left(\bar\xi_{\tau_\ell}^\varepsilon\left(\phi^{(\ell)}\right)\right)_{\ell\leq L}$ (derived in Proposition 3.3 below) will imply the convergence in law of $\left(\xi_{\tau_\ell}^\varepsilon\left(\phi^{(\ell)}\right)\right)_{\ell\leq L}$, i.e. Proposition 3.1.

Proof of Lemma 3.2. Recall (see Proposition A.1 in [11]) that for any ε small enough, the following holds under the equilibrium measure for any function h

$$\mathbb{E}_\varepsilon\Big(\big(\xi_\tau^\varepsilon(h)\big)^q\Big) \leq C_q\|h\|^q_{L^q_{\mathcal{M}}}\,, \tag{3.6}$$

with $1\leq q<\infty$ and where $L^q_{\mathcal{M}}$ is defined as in (2.3). Since for $R\geq 1$

$$\begin{aligned}
&\left\|\varphi\cdot v\left(\chi\left(\frac{|v|^2}{R}\right)-1\right)\right\|^q_{L^q_{\mathcal{M}}} \leq C\|\varphi\|^q_{L^q(\mathbb{T}^d)}e^{-R/4}\\
&\left\|\psi\left(\frac{|v|^2}{d+2}-1\right)\left(\chi\left(\frac{|v|^2}{R}\right)-1\right)\right\|^q_{L^q_{\mathcal{M}}} \leq C\|\psi\|^q_{L^q(\mathbb{T}^d)}e^{-R/4}\,,
\end{aligned} \tag{3.7}$$

we find (3.4). For the same reason (3.5) holds. This completes Lemma 3.2. $\square$

3.2. Covariance of the Hydrodynamic Fields

Proposition 3.3. *For arbitrary times $\tau_1, \dots, \tau_L$ and test functions $\phi^{(1)} = (\varphi^{(1)}, \psi^{(1)}), \dots$ and $\phi^{(L)} = (\varphi^{(L)}, \psi^{(L)})$ chosen as in (1.18), the time marginals $\big(\bar\xi^{\varepsilon}_{\tau_\ell}(\phi^{(\ell)})\big)_{\ell\le L}$ converge in law to the limiting process $\big(\mathcal{U}_{\tau_\ell}(\varphi^{(\ell)}), \Theta_{\tau_\ell}(\psi^{(\ell)})\big)_{\ell\le L}$ as μ_ε tends to infinity.*

Combined with the approximation Lemma 3.2, this completes the proof of Proposition 3.1. The proof of Proposition 3.3 is split into two parts, first a control of the limiting covariance and then the derivation of Wick's rule to prove that the limiting process is Gaussian.

Step 1. Control of the covariance. Let us define the hydrodynamic, non-oscillating projections

$$\begin{aligned} g^0(x,v) &:= \left(u_0(x)\cdot v + \theta_0(x)\frac{|v|^2-(d+2)}{2}\right), \\ h(x,v) &:= \left(\varphi(x)\cdot v + \psi(x)\left(\frac{|v|^2}{d+2}-1\right)\right), \end{aligned} \tag{3.8}$$

for some smooth divergence free vector fields u_0, φ, and some smooth functions θ_0, ψ. The scaling in g^0, h has been tuned asymmetrically so that the initial covariance is given by

$$\mathbb{E}_\varepsilon\left[\bar\xi^\varepsilon_0(\phi_0)\bar\xi^\varepsilon_0(\phi)\right] \longrightarrow \int (u_0\cdot\varphi + \theta_0\psi)\mathrm{d}x\,, \quad \mu_\varepsilon\to\infty\,.$$

We are going to study the covariance of the joint process $\bar\xi^\varepsilon_\tau$ by applying Theorem 2.1 with

$$\bar g^0(x,v) := g^0(x,v)\chi\left(\frac{|v|^2}{R}\right), \quad \bar h(x,v) = h(x,v)\chi\left(\frac{|v|^2}{R}\right). \tag{3.9}$$

Setting

$$\phi_0 := \left(u_0, \frac{d+2}{2}\theta_0\right), \quad \phi := (\varphi,\psi)\,,$$

we plug the bounds (3.3) on the test functions into the estimate (2.4) of Theorem 2.1, and recalling the definition (3.1) of the truncated rescaled fluctuation field, we obtain that for any $T>0$ such that $(T/\alpha^2)^{3/2} \ll (\log\log\mu_\varepsilon)^{1/4}$,

$$\begin{aligned} &\sup_{t\in[0,T]}\left|\mathbb{E}_\varepsilon\left[\bar\xi^\varepsilon_0(\phi_0)\bar\xi^\varepsilon_\tau(\phi)\right] - \int \mathcal{M}\tilde g_\alpha(t)\bar h\mathrm{d}x\mathrm{d}v\right| \\ &\qquad \le CR\|\phi_0\|_{W^{1,\infty}}\|\phi\|_{W^{1,\infty}}\left(\frac{CT}{\alpha^2}\right)^{3/2}(\log\log\mu_\varepsilon)^{-1/4}, \end{aligned} \tag{3.10}$$

where $\tilde g_\alpha$ is the solution to the time-rescaled equation

$$\alpha\partial_\tau\tilde g_\alpha + v\cdot\nabla_x\tilde g_\alpha + \frac{1}{\alpha}\mathcal{L}\tilde g_\alpha = 0\,, \qquad \tilde g_{\alpha|\tau=0} = \bar g^0\,. \tag{3.11}$$

To conclude the convergence of the covariance as $\alpha\to0$, we just need to identify the limit of $\int \mathcal{M}\tilde g_\alpha(\tau)\bar h\mathrm{d}x\mathrm{d}v$.

The starting point for the study of hydrodynamic limits of the linearized Boltzmann equation (3.11) is the scaled energy inequality

$$\frac{1}{2}\|\tilde{g}_\alpha(\tau)\|^2_{L^2(\mathcal{M}dvdx)} + \frac{1}{\alpha^2}\int_0^\tau \int \tilde{g}_\alpha \mathcal{L}\tilde{g}_\alpha(\tau')\mathcal{M}dvdxd\tau' \le \frac{1}{2}\|\bar{g}^0\|^2_{L^2(\mathcal{M}dvdx)}. \tag{3.12}$$

Recall (see [18,19]) that the linearized collision operator $\mathcal{L}$ with hard sphere cross section defined by (1.10) is a nonnegative unbounded self-adjoint operator on $L^2(\mathcal{M}\mathrm{d}v)$ with domain

$$\mathcal{D}(\mathcal{L}) = L^2\left(\mathbb{R}^d; (1+|v|)\mathcal{M}\mathrm{d}v\right)$$

and nullspace

$$\mathrm{Ker}(\mathcal{L}) = \mathrm{span}\left\{1, v_1, \dots, v_d, |v|^2\right\}.$$

In particular we recover from (3.12) the uniform L^2 bound

$$\|\tilde{g}_\alpha(\tau)\|_{L^2(\mathcal{M}\mathrm{d}v\mathrm{d}x)} \le \|\bar{g}^0\|_{L^2(\mathcal{M}\mathrm{d}v\mathrm{d}x)} \le \|g^0\|_{L^2(\mathcal{M}\mathrm{d}v\mathrm{d}x)}.$$

This bound implies that there is $g \in L^\infty_\tau(L^2(\mathcal{M}\mathrm{d}v\mathrm{d}x))$ such that, up to extraction of a subsequence,

$$\tilde{g}_\alpha \rightharpoonup g \text{ weakly in } L^2_{\mathrm{loc}}(\mathrm{d}\tau, L^2(\mathcal{M}\mathrm{d}v\mathrm{d}x)). \tag{3.13}$$

Moreover the following coercivity estimate holds : there exists $C > 0$ such that, for each g in $\mathcal{D}(\mathcal{L}) \cap (\mathrm{Ker}(\mathcal{L}))^\perp$

$$\int g\mathcal{L}g(v)\mathcal{M}(v)\mathrm{d}v \ge C\|g\|^2_{L^2((1+|v|)\mathcal{M}\mathrm{d}v)}. \tag{3.14}$$

The dissipation thus further provides

$$\|\tilde{g}_\alpha - \Pi\tilde{g}_\alpha\|_{L^2((1+|v|)\mathcal{M}\mathrm{d}v\mathrm{d}x\mathrm{d}t)} = O(\alpha),$$

where Π denotes the orthogonal projection onto $\mathrm{Ker}(\mathcal{L})$ in $L^2(\mathcal{M}\mathrm{d}v\mathrm{d}x)$. We deduce from the previous estimate that

$$g(\tau,x,v) = \Pi g(\tau,x,v) \equiv \rho(\tau,x) + u(\tau,x)\cdot v + \theta(\tau,x)\frac{|v|^2-d}{2}. \tag{3.15}$$

It remains to compute the equations on ρ, u and θ. Denoting $\langle g\rangle := \int g\mathcal{M}\mathrm{d}v$ and recalling (3.11), the moment equations state

$$\begin{aligned}
&\alpha\partial_\tau\langle\tilde{g}_\alpha\rangle + \nabla_x\cdot\langle\tilde{g}_\alpha v\rangle = 0,\\
&\alpha\partial_\tau\langle\tilde{g}_\alpha v\rangle + \nabla_x\cdot\langle\tilde{g}_\alpha v\otimes v\rangle = 0,\\
&\alpha\partial_\tau\langle\tilde{g}_\alpha|v|^2\rangle + \nabla_x\cdot\langle\tilde{g}_\alpha v|v|^2\rangle = 0.
\end{aligned}$$

Using (3.13) and (3.15) we deduce from the first two equations that

$$\nabla_x\cdot u = 0, \quad \nabla_x(\rho+\theta) = 0, \tag{3.16}$$

referred to as the incompressibility and Boussinesq constraints. We thus have

$$g(\tau,x,v) = u(\tau,x)\cdot v + \theta(\tau,x)\frac{|v|^2-(d+2)}{2}, \quad \nabla_x\cdot u = 0. \tag{3.17}$$

Note that, up to the cut-off in v which can be removed with a small error thanks to (3.7), the test function $\bar h$ is in the kernel of the acoustic operator. It follows that we only need to characterize the mean motion, namely derive the equations for $P\langle \tilde g_\alpha v\rangle$ and $\langle \tilde g_\alpha(|v|^2-d-2)\rangle$:

$$\begin{aligned}
&\partial_\tau P\langle \tilde g_\alpha v\rangle + \frac{1}{\alpha}P\nabla_x\cdot\left\langle \tilde g_\alpha\left(v\otimes v-\frac{1}{d}|v|^2\mathrm{Id}\right)\right\rangle = 0\,,\\
&\partial_\tau \frac{1}{d+2}\left\langle \tilde g_\alpha\left(|v|^2-d-2\right)\right\rangle + \frac{1}{\alpha}\nabla_x\cdot\langle \tilde g_\alpha\frac{1}{d+2}v\Big(|v|^2-d-2\Big)\rangle = 0,
\end{aligned}$$

where we recall that P is the Leray projection on divergence free vector fields. Define the kinetic momentum flux $A(v):=v\otimes v-\dfrac{1}{d}|v|^2\mathrm{Id}$ and the kinetic energy flux $B(v):=\dfrac{1}{2}v(|v|^2-d-2)$. As A,B belong to $(\mathrm{Ker}\,\mathcal{L})^\perp$, and $\mathcal{L}$ is a Fredholm operator, there exist pseudo-inverses $\tilde A,\tilde B$ in $(\mathrm{Ker}\,\mathcal{L})^\perp$ such that $A=\mathcal{L}\tilde A$ and $B=\mathcal{L}\tilde B$. Then,

$$\begin{aligned}
&\partial_\tau P\langle \tilde g_\alpha v\rangle + \frac{1}{\alpha}P\nabla_x\cdot\langle(\mathcal{L}\tilde g_\alpha)\tilde A\rangle = 0\,,\\
&\frac{1}{d+2}\partial_\tau\langle \tilde g_\alpha(|v|^2-d-2)\rangle + \frac{1}{\alpha}\frac{2}{d+2}\nabla_x\cdot\langle(\mathcal{L}\tilde g_\alpha)\tilde B\rangle = 0\,.
\end{aligned}$$

Using the equation

$$\frac{1}{\alpha}\mathcal{L}\tilde g_\alpha = -v\cdot\nabla_x\tilde g_\alpha - \alpha\partial_\tau\tilde g_\alpha \tag{3.18}$$

we get

$$\begin{aligned}
&\partial_\tau P\langle \tilde g_\alpha v\rangle - P\nabla_x\cdot\langle(v\cdot\nabla_x+\alpha\partial_\tau)\tilde g_\alpha\tilde A\rangle = 0\,,\\
&\frac{1}{d+2}\partial_\tau\langle \tilde g_\alpha(|v|^2-d-2)\rangle - \frac{2}{d+2}\,\nabla_x\cdot\langle(v\cdot\nabla_x+\alpha\partial_\tau)\tilde g_\alpha\tilde B\rangle = 0\,.
\end{aligned} \tag{3.19}$$

Then, plugging the Ansatz (3.15), and taking limits in the sense of distributions, we get the Stokes–Fourier equations

$$\begin{aligned}
&\partial_\tau u - \nu\Delta_x u = 0\,,\quad \nabla_x\cdot u = 0\,,\\
&\partial_\tau\theta - \kappa\Delta_x\theta = 0\,,
\end{aligned}$$

with initial data as in (1.17)

$$\begin{aligned}
&u_{|\tau=0}(x) := P\int g_0(x,v)v\mathcal{M}(v)\,\mathrm{d}v\,,\\
&\theta_{|\tau=0}(x) := \int g_0(x,v)\left(\frac{|v|^2}{d+2}-1\right)\mathcal{M}(v)\,\mathrm{d}v,
\end{aligned}$$

and where the diffusion coefficients are given by

$$\nu := \frac{1}{(d-1)(d+2)}\langle A:\tilde A\rangle \quad\text{and}\quad \kappa := \frac{2}{d(d+2)}\langle B\cdot\tilde B\rangle\,. \tag{3.20}$$

We therefore end up with the following convergence as $\alpha\to 0$

$$\int \mathcal{M}\tilde g_\alpha(\tau)\bar h\mathrm{d}x\mathrm{d}v \longrightarrow \int (u(\tau)\cdot\varphi+\theta(\tau)\psi)\mathrm{d}x\,. \tag{3.21}$$

Returning to (3.10), we have proved that

$$\sup_{\tau\in[0,T]} \mathbb{E}_\varepsilon\left[\bar\xi_0^\varepsilon(\phi_0)\bar\xi_\tau^\varepsilon(\phi)\right] \longrightarrow \int (u(\tau)\cdot\varphi + \theta(\tau)\psi)\mathrm{d}x\,, \quad \mu_\varepsilon\to\infty\,. \tag{3.22}$$

Remark 3.4. Since the initial data g^0 is well-prepared, both the purely kinetic component and the fast oscillating acoustic waves are negligible, so the convergence of $\tilde g_\alpha$ can be shown actually to hold in strong sense. Using energy methods, it is even possible to obtain a rate of convergence for (3.21).

Step 2. Wick's rule Consider p times $\tau_1,\dots,\tau_p$, possibly repeated. Thanks to the cut-off (3.3), we can apply Theorem 2.2 to obtain

$$\begin{aligned}&\left|\mathbb{E}_\varepsilon\Big[\bar\xi^\varepsilon_{\tau_1}\big(\phi^{(1)}\big)\ \dots\ \bar\xi^\varepsilon_{\tau_p}\big(\phi^{(p)}\big)\Big] - \sum_{\eta\in\mathfrak{S}_p^{\rm pairs}}\ \prod_{\{i,j\}\in\eta}\mathbb{E}_\varepsilon\left[\bar\xi^\varepsilon_{\tau_i}\big(\phi^{(i)}\big)\bar\xi^\varepsilon_{\tau_j}\big(\phi^{(j)}\big)\right]\right|\\ &\qquad\le C_pR^p\prod_{i=1}^p\|\phi^{(i)}\|_{L^\infty}\Big(\frac{CT}{\alpha^2}\Big)^{(2p-1)/2}(\log\log\mu_\varepsilon)^{-1/4}\,.\end{aligned}\tag{3.23}$$

With the scaling condition (3.2), we get that the right-hand side converges to 0 as $\mu_\varepsilon\to\infty$ which implies the asymptotic pairing of the moments of $\bar\xi^\varepsilon_\tau$. Since the limiting covariance is characterized by (3.22), this completes Proposition 3.3. □

4. Tightness of Hydrodynamic Fields on Diffusive Time Scales

Let us first introduce for any $k\in\mathbb{Z}$ the Sobolev space $\mathbb{H}^k$ in $\mathbb{T}^d$ with the norm

$$\|F\|_k^2 := \sum_{j\in\mathbb{Z}^d}\big(1+|j|^2\big)^k\,|\hat F_j|^2, \tag{4.1}$$

where $(\hat F_j)$ stand for the Fourier coefficients of F.

Proposition 4.1. *There exists $k>0$ such that, in the diffusive limit*

$$\mu_\varepsilon\to\infty, \alpha\to 0, \qquad \textit{with}\quad \mu_\varepsilon\varepsilon^{d-1}=\alpha^{-1}\le\log\log\log\mu_\varepsilon,$$

the fluctuation field $(\xi^\varepsilon_\tau)_{\tau\ge0}$ defined by (1.19) is tight in the Skorokhod space $D\big([0,T],\mathbb{H}^{-k}\big)$. More precisely,

$$\begin{aligned}&\lim_{\delta\to0^+}\lim_{\mu_\varepsilon\to\infty}\mathbb{P}_\varepsilon\Big[\sup_{\substack{|\sigma-\tau|\le\delta\\ s,\tau\in[0,T]}}\big\|\xi^\varepsilon_\tau-\xi^\varepsilon_\sigma\big\|_{-k}\ge\delta'\Big]=0\,, \qquad \forall\delta'>0\,,\\ &\lim_{A\to\infty}\lim_{\mu_\varepsilon\to\infty}\mathbb{P}_\varepsilon\Big[\sup_{\tau\in[0,T]}\big\|\xi^\varepsilon_\tau\big\|_{-k}\ge A\Big]=0\,.\end{aligned}\tag{4.2}$$

The tightness property for kinetic times relies on the Garsia–Rodemich–Rumsey inequality on the modulus of continuity of a function $\varphi_\tau:[0,T]\to\mathbb{R}$, which we recall ([27]): for $b\ge4$

$$\sup_{\substack{0\le\sigma,\tau\le T\\|\tau-\sigma|\le\delta}}\big|\varphi_\tau-\varphi_\sigma\big|\le C\left(\int_0^T\int_0^T \mathrm{d}\sigma\mathrm{d}\tau\,\frac{|\varphi_\tau-\varphi_\sigma|^b}{|\tau-\sigma|^\gamma}\right)^{1/b}\delta^{\frac{\gamma-2}{b}}\,, \qquad \gamma\in]2,3[\,. \tag{4.3}$$

Because of collisions in the Newtonian dynamics, the fluctuation field ξ^ε has jumps and this inequality does not apply directly. We therefore start by stating a modified inequality, whose proof is a slight adaptation of [27] which can be found in [10] (see Proposition 6.2.4).

Proposition 4.2. *Let $F : [0,T] \to \mathbb{R}$ be a given function and define for $a > 0$, $b \geq 4$*

$$B_a(F) := \int_0^T \int_0^T d\sigma d\tau \, \frac{|F_\tau - F_\sigma|^b}{|\tau - \sigma|^\gamma} \mathbf{1}_{|\tau-\sigma|>a} \,, \qquad \gamma \in]2,3[\,. \tag{4.4}$$

Then the modulus of continuity of F is controlled by

$$\sup_{\substack{0\leq\sigma,\tau\leq T\\ |\tau-\sigma|\leq\delta}} \big|F_\tau - F_\sigma\big| \leq 2 \sup_{\substack{0\leq\sigma,\tau\leq T\\ |\tau-\sigma|\leq 2a}} \big|F_\tau - F_\sigma\big| \;+\; C B_a(F)^{\frac{1}{b}} \, \delta^{\frac{\gamma-2}{b}} \,. \tag{4.5}$$

Proof of Proposition 4.1. To prove the tightness of the joint process $(\xi^\varepsilon_\tau)_{\tau\geq 0}$ in $D([0,T], \mathbb{H}^{-k})$ for some k large enough, we shall tune the parameter a, introduced in the statement of Proposition 4.2, as a small fraction of the kinetic time, i.e. $a \ll \alpha^2$ in the diffusive scaling. More precisely, we shall use (4.4) with the parameters

$$b = 6 \,, \quad \gamma = 7/3 \,, \quad a = (\log\log\mu_\varepsilon)^{-1/10}, \quad \alpha = (\log\log\log\mu_\varepsilon)^{-1}. \tag{4.6}$$

We deduce from (4.5) that, for arbitrary $\delta' > 0$,

$$\begin{aligned} &\mathbb{P}_\varepsilon \left(\sup_{\substack{0\leq\tau,\sigma\leq T\\ |\tau-\sigma|\leq\delta}} \big\|\xi^\varepsilon_\tau - \xi^\varepsilon_\sigma\big\|^2_{-k} \geq \delta' \right) \\ &\quad\leq \mathbb{P}_\varepsilon \left(\sum_j \frac{C^2 B_a\big(\xi^\varepsilon(\phi_j)\big)^{1/3}}{(1+|j|^2)^k} \delta^{\frac{\gamma-2}{3}} \geq \frac{\delta'}{4} \right) \\ &\qquad + \mathbb{P}_\varepsilon \left(\sum_j \frac{4}{(1+|j|^2)^k} \sup_{\substack{|\sigma-\tau|\leq 2a\\ \sigma,\tau\in[0,T]}} \big|\xi^\varepsilon_\tau(\phi_j) - \xi^\varepsilon_\sigma(\phi_j)\big|^2 \geq \frac{\delta'}{4} \right), \end{aligned} \tag{4.7}$$

where $\phi_j(x) = \exp(2i\pi j \cdot x)$ are the Fourier modes used to define the norm (4.1). Since $a \ll \alpha^2$, the two events in the right-hand side of inequality (4.7) control different time scales and their probabilities have to be estimated by different methods :

- for time increments $|\sigma - \tau| \geq a$, by a control on moments using the comparison with the limit process;
- for small time increments $|\sigma - \tau| \leq 2a$, by reducing to the estimates on the kinetic times obtained in [10] (see Proposition 6.2.3). To do this, additional cut-off estimates to control divergences at large velocities are necessary.

Step 1. Control of the short hydrodynamic increments.

We are first going to prove that

$$\lim_{\delta\to 0}\lim_{\mu_\varepsilon\to\infty}\mathbb{P}_\varepsilon\left(\sum_j \frac{CB_a\big(\xi^\varepsilon(\phi_j)\big)^{1/3}}{(1+|j|^2)^k}\delta^{\frac{\gamma-2}{3}} \geq \frac{\delta'}{4}\right) = 0\,. \tag{4.8}$$

Assume that the following bound holds

$$\mathbb{E}_\varepsilon\Big(B_a\big(\xi^\varepsilon(\phi)\big)\Big) \leq C\,\|\phi\|^6_{W^{2,\infty}}. \tag{4.9}$$

Since for the Fourier basis $\|\phi_j\|_{W^{2,\infty}} \leq C|j|^2$, we deduce from (4.9) that for $k > d/2+2$, (4.8) follows from a Markov inequality as $\gamma > 2$

$$\begin{aligned}\mathbb{P}_\varepsilon&\left(\sum_j \frac{C^2B_a\big(\xi^\varepsilon(\phi_j)\big)^{1/3}}{(1+|j|^2)^k}\delta^{\frac{\gamma-2}{3}} \geq \frac{\delta'}{4}\right)\\ &\leq C\frac{\delta^{\frac{\gamma-2}{3}}}{\delta'}\sum_j \frac{1}{(1+|j|^2)^k}\mathbb{E}_\varepsilon\left(B_a\big(\xi^\varepsilon(\phi_j)\big)\right)^{1/3}.\end{aligned}$$

We turn now to the proof of (4.9). As $\gamma = 7/3$, this will be a consequence of the following inequality

$$\forall \tau,\sigma\in[0,T]\,,\qquad \mathbb{E}_\varepsilon\left[\Big(\xi^\varepsilon_\tau(\phi)-\xi^\varepsilon_\sigma(\phi)\Big)^6\right]\mathbf{1}_{|\tau-\sigma|\geq a} \leq C\,\|\phi\|^6_{W^{2,\infty}}\,|\tau-\sigma|^{3/2}\,. \tag{4.10}$$

Applying Lemma 3.2, it is enough to derive (4.10) for the truncated process $\bar\xi^\varepsilon$ with cut-off $R = \log\log\log\mu_\varepsilon$ because

$$\forall\tau\leq T,\qquad \mathbb{E}_\varepsilon\left[\big(\xi^\varepsilon_\tau(\phi)-\bar\xi^\varepsilon_\tau(\phi)\big)^6\right] \leq C\|\phi\|^6_{L^6(\mathbb{T}^d)}e^{-R/4} \leq C\|\phi\|^6_{L^6(\mathbb{T}^d)}\,a^2\,,$$

with a defined in (4.6).

Our starting point is the asymptotic factorization (3.23) of the moments leading to the following formula for the time increments

$$\begin{aligned}&\left|\mathbb{E}_\varepsilon\left[\Big(\bar\xi^\varepsilon_\tau(\phi)-\bar\xi^\varepsilon_\sigma(\phi)\Big)^6\right] - 15\,\mathbb{E}_\varepsilon\left[\Big(\bar\xi^\varepsilon_\tau(\phi)-\bar\xi^\varepsilon_\sigma(\phi)\Big)^2\right]^3\right|\\ &\qquad\leq C_6R^6\,\|\phi\|^6_{L^\infty}\left(\frac{CT}{\alpha^2}\right)^{11/2}(\log\log\mu_\varepsilon)^{-1/4} \leq C\|\phi\|^6_{L^6(\mathbb{T}^d)}\,a^2,\end{aligned} \tag{4.11}$$

uniformly in $\tau,\sigma\in[0,T]$, with our choice of scaling (3.2).

Next we are going to use that, by (3.10), the covariance is well approximated by the solution to the linearized Boltzmann equation (3.11). Denoting by $\tilde g_\alpha$ the solution of the linearized Boltzmann equation (3.11) with truncated initial data (3.9), we get that

$$\begin{aligned}\sup_{\sigma,\tau\in[0,T]}&\left|\mathbb{E}_\varepsilon\left[\Big(\bar\xi^\varepsilon_\tau(\phi)-\bar\xi^\varepsilon_\sigma(\phi)\Big)^2\right] - 2\int\mathcal{M}(\bar g^0-\tilde g_\alpha(\tau-\sigma))\,g^0\mathrm{d}x\mathrm{d}v\right|\\ &\leq CR^2\|\phi\|^2_{W^{1,\infty}}\left(\frac{CT^3}{\alpha^6}\right)^{1/2}(\log\log\mu_\varepsilon)^{-1/4}\end{aligned}$$

$$+ C\|\phi\|^2_{L^2} e^{-R/4} \le C\,\|\phi\|^2_{W^{1,\infty}}\ a^2\,, \tag{4.12}$$

using the time invariance of the equilibrium measure and the control (3.7) to remove the velocity cutoff on (one of) the initial data $\bar{g}^0$ in the integral. From (3.19) we have

$$\begin{aligned}
&\partial_\tau \Big(P\langle \tilde{g}_\alpha v\rangle - \alpha P\nabla_x \cdot \langle \tilde{g}_\alpha \tilde{A}\rangle \Big) - P\nabla_x \cdot \langle v\cdot \nabla_x(\tilde{g}_\alpha \tilde{A})\rangle = 0\,,\\
&\frac{1}{d+2}\partial_\tau \Big(\langle \tilde{g}_\alpha(|v|^2 - d - 2)\rangle - 2\alpha\nabla_x \cdot \langle \tilde{g}_\alpha \tilde{B}\rangle \Big) - \frac{2}{d+2}\nabla_x \cdot \langle v\cdot \nabla_x(\tilde{g}_\alpha \tilde{B})\rangle = 0,
\end{aligned}$$

so thanks to the uniform $L^\infty_\tau(L^2(\mathcal{M}\mathrm{d}x\mathrm{d}v))$ bound on $\tilde{g}_\alpha$ we deduce that

$$\begin{aligned}
&P\,\langle \tilde{g}_\alpha v\rangle - \alpha P\nabla_x \cdot \langle \tilde{g}_\alpha \tilde{A}\rangle \text{ is uniformly bounded in } W^{1,\infty}_\tau(\mathbb{H}^{-2}),\\
&\left\langle \tilde{g}_\alpha \frac{|v|^2 - (d+2)}{2} \right\rangle - \alpha\nabla_x \cdot \langle \tilde{g}_\alpha \tilde{B}\rangle \text{ is uniformly bounded in } W^{1,\infty}_\tau(\mathbb{H}^{-2})\,.
\end{aligned} \tag{4.13}$$

We then have to control the time regularity of the $O(\alpha)$ terms in (4.13). From (3.12) and (3.14), we get uniform bounds on $\tilde{g}_\alpha$ in $L^\infty_\tau(L^2(\mathcal{M}\mathrm{d}x\mathrm{d}v)) \cap L^2_{loc,\tau}(l^2(\mathcal{M}(1+|v|)dxdv))$. Then, using the generic notation $\mathfrak{p}$ for $\tilde{A}$ and $\tilde{B}$, we have $\mathfrak{p} \in L^2(\mathcal{M}(1+|v|dv)$ so that

$$\forall \tau \in [0,T], \qquad \|\nabla_x \langle \tilde{g}_\alpha \mathfrak{p}(v)\rangle\|_{\mathbb{H}^{-1}} \le C. \tag{4.14}$$

Applying the kinetic equation (3.11),

$$\partial_\tau \langle \tilde{g}_\alpha \mathfrak{p}(v)\rangle + \frac{1}{\alpha}\nabla_x \cdot \langle \tilde{g}_\alpha \mathfrak{p}(v) v\rangle + \frac{1}{\alpha^2}\langle \mathcal{L}\tilde{g}_\alpha \mathfrak{p}(v)\rangle = 0$$

we also obtain that $\Rightarrow \|\langle \tilde{g}_\alpha(\tau)\mathfrak{p}(v)\rangle - \langle \tilde{g}_\alpha(\sigma)\mathfrak{p}(v)\rangle\|_{\mathbb{H}^{-1}} \le + \|\langle \tilde{g}_\alpha(\tau)\mathfrak{p}(v)\rangle -$

$$\langle \tilde{g}_\alpha(\sigma)\mathfrak{p}(v)\rangle\|_{\mathbb{H}^{-1}} \le C\frac{|\tau-\sigma|}{\alpha^2}^{1/2} + C\frac{|\tau-\sigma|}{\alpha^2}.$$

We conclude that

$$\begin{aligned}
&\Big\|\alpha P\nabla_x \cdot \langle \tilde{g}_\alpha(\tau)\tilde{A}\rangle - \alpha P\nabla_x \cdot \langle \tilde{g}_\alpha(\sigma)\tilde{A}\rangle\Big\|_{\mathbb{H}^{-2}} \le C\min\left(\alpha, |\tau-\alpha|^{1/2} + \frac{|\tau-\sigma|}{\alpha}\right) \le C|\tau-\sigma|^{1/2},\\
&\Big\|\alpha \nabla_x \cdot \langle \tilde{g}_\alpha(\tau)\tilde{B}\rangle - \alpha \nabla_x \cdot \langle \tilde{g}_\alpha(\sigma)\tilde{B}\rangle\Big\|_{\mathbb{H}^{-2}} \le C\min\left(\alpha, |\tau-\alpha|^{1/2} + \frac{|\tau-\sigma|}{\alpha}\right) \le C|\tau-\sigma|^{1/2}\,.
\end{aligned}$$

Therefore, applying (4.13), we deduce that the bulk velocity $P\langle \tilde{g}_\alpha v\rangle$ and temperature $\langle \tilde{g}_\alpha \frac{|v|^2-(d+2)}{d+2}\rangle$ are uniformly bounded in $C^{1/2}_\tau(\mathbb{H}^{-2}_x)$. Since the initial data g^0 is well prepared (see (3.8)), we deduce that the term involving the linearized equation in (4.12) is controlled by

$$
\begin{aligned}
&\Big|\int \mathcal{M}(\bar g^0 - \tilde g_\alpha(\tau-\sigma))\, g^0 \mathrm{d}x\mathrm{d}v\Big| \\
&\quad \le \Big|\int \Big(P\langle \tilde g_\alpha(\tau-\sigma)v\rangle - P\langle \bar g_0 v\rangle\Big)\cdot u_0 \mathrm{d}x\Big| \\
&\qquad + \frac{d+2}{2}\Big|\int \Big(\Big\langle \tilde g_\alpha(\tau-\sigma)\frac{|v|^2-(d+2)}{2}\Big\rangle - \Big\langle \bar g_0\frac{|v|^2-(d+2)}{2}\Big\rangle\Big)\theta_0 \mathrm{d}x\Big| \\
&\quad \le C\,\|\phi\|^2_{W^{2,\infty}}\,|\tau-\sigma|^{1/2}.
\end{aligned}
$$

Combining (4.11)-(4.12) and the time regularity of the covariance, we get that for $|\tau-\sigma|\ge a$

$$
\mathbb{E}_\varepsilon\left[\Big(\xi^\varepsilon_\tau(\phi)-\xi^\varepsilon_\sigma(\phi)\Big)^6\right]1_{|\tau-\sigma|\ge a} \le C\,\|\phi\|^6_{W^{2,\infty}}\,|\tau-\sigma|^{3/2}.
$$

This completes the proof of Inequality (4.10).

Step 2. Control of the very short kinetic times.

Finally, it remains to control the second term in (4.7). By splitting the time interval $[0,T]$ into intervals with kinetic time length scale α^2, the estimate can be reduced, by using the invariant measure and an union bound, to

$$
\mathbb{P}_\varepsilon\left(\sum_j \frac{4}{(1+|j|^2)^k}\sup_{\substack{|\sigma-\tau|\le 2a\\ \sigma,\tau\in[0,T]}}\big|\xi^\varepsilon_\tau(\phi_j)-\xi^\varepsilon_\sigma(\phi_j)\big|^2 \ge \frac{\delta'}{4}\right) \le \frac{T}{\alpha^2}\mathbb{P}_\varepsilon(\mathcal{A}), \tag{4.15}
$$

with the notation

$$
\mathcal{A} := \left\{\sum_j \frac{4}{(1+|j|^2)^k}\sup_{\substack{|\sigma-\tau|\le 2a\\ \sigma,\tau\in[0,\alpha^2]}}\big|\xi^\varepsilon_\tau(\phi_j)-\xi^\varepsilon_\sigma(\phi_j)\big|^2 \ge \frac{\delta'}{4}\right\}. \tag{4.16}
$$

Recalling that $a \ll \alpha^2$, we are going to show that

$$
\lim_{\mu_\varepsilon\to\infty}\frac{1}{\alpha^2}\mathbb{P}_\varepsilon(\mathcal{A}) = 0, \tag{4.17}
$$

which is essentially the outcome of Proposition 6.2.3 in [10], however the proof cannot be applied directly in our context and we explain below the necessary adjustments.

First of all, the test functions are now unbounded in v (contrary to the Fourier–Hermite modes). Thus an energy cut-off is necessary. For technical reasons, we are going to use a larger truncation parameter $\tilde R = (\log\mu_\varepsilon)^2$ instead of $R = \alpha^{-1}$ introduced in (3.2). The corresponding truncated process is defined as in (3.1) and denoted by $(\tilde\xi^\varepsilon_\tau)_{\tau\ge 0}$. We are going to check that with high probability both processes coincide because all the velocities remain smaller than $\sqrt{\tilde R}$

$$
\lim_{\mu_\varepsilon\to\infty}\frac{1}{\alpha^2}\mathbb{P}_\varepsilon\left(\exists i,\quad \sup_{t\le\alpha}\big|\mathbf{v}^\varepsilon_i(t)\big| > \sqrt{\tilde R}\right) = 0. \tag{4.18}
$$

This can be deduced from a result of [11] as follows. Fix $n = 4d, \eta = \varepsilon^{1-\frac{1}{2d}}$ and call *microscopic cluster of size* n a set $\mathcal{G}$ of n particle configurations in $\mathbb{T}^d \times \mathbb{R}^d$ such that $(z, z') \in \mathcal{G} \times \mathcal{G}$ if and only if there are $z_1 = z, z_2, \dots, z_\ell = z'$ in $\mathcal{G}$ such that

$$|x_i - x_{i+1}| \leq 3\sqrt{\tilde{R}}\,\eta, \quad \forall 1 \leq i \leq \ell - 1\,.$$

Let Υ_N^ε be the set of initial configurations $\mathbf{Z}_N^{\varepsilon 0} \in \mathcal{D}_N^\varepsilon$ such that for any integer $1 \leq k \leq \frac{\alpha}{\eta}$, the configuration at time $k\eta$ satisfies

$$\forall 1 \leq j \leq N, \qquad |v_j| \leq \frac{\sqrt{\tilde{R}}}{n}\,, \tag{4.19}$$

and any microscopic cluster of particles is of size at most n. Adapting to our framework the proof of Proposition 2.7 of [11] implies that

$$\mathbb{P}_\varepsilon\big({}^c\Upsilon_{\mathcal{N}}^\varepsilon\big) \leq \frac{1}{\alpha^n}\,\varepsilon^d. \tag{4.20}$$

We check that for any configuration in $\Upsilon_{\mathcal{N}}^\varepsilon$, the velocities are bounded from above by $\sqrt{\tilde{R}}$ during the kinetic time interval $[0, \alpha]$. Indeed, at each intermediate time $k\eta$, the velocities of configurations in $\Upsilon_{\mathcal{N}}^\varepsilon$, are smaller than $\frac{\sqrt{\tilde{R}}}{n}$ by (4.19). Furthermore the clusters are all of size less than n and in the time interval $[k\,\eta, (k+1)\eta]$ particles within a cluster cannot interact with particles in other clusters. As the total kinetic energy of a finite number of particles is preserved by the hard sphere dynamics, the velocity of each particle will remain less than $\sqrt{\tilde{R}}$. Thus (4.18) is implied by (4.20).

We are now in position to complete the proof of (4.17). Thanks to (4.18), it is enough to replace the event $\mathcal{A}$ by the similar event $\tilde{\mathcal{A}}$ for the process $(\tilde{\xi}_\tau^\varepsilon)_{\tau \geq 0}$. It thus remains to prove

$$\lim_{\mu_\varepsilon \to \infty} \frac{1}{\alpha^2}\mathbb{P}_\varepsilon\left(\tilde{\mathcal{A}}\right) = 0. \tag{4.21}$$

The statement of Proposition 6.2.3 from [10] is not precise enough to conclude directly mainly due to the diverging prefactor $\frac{1}{\alpha^2}$. However all the required estimates can be found in [10] and we are going to detail the relevant parts of the argument.

We proceed as in (4.7) and introduce an additional time cut-off $\mu_\varepsilon^{-7/3}$ instead of a to filter the very small scales

$$\begin{aligned}
\frac{1}{\alpha^2}\mathbb{P}_\varepsilon\left(\tilde{\mathcal{A}}\right) &= \frac{1}{\alpha^2}\mathbb{P}_\varepsilon\left(\sup_{\substack{0 \leq \tau, \sigma \leq \alpha^2 \\ |\tau - \sigma| \leq 2a}} \left\|\tilde{\xi}_\tau^\varepsilon - \tilde{\xi}_\sigma^\varepsilon\right\|_{-k}^2 \geq \frac{\delta'}{16}\right) \\
&\leq \frac{1}{\alpha^2}\mathbb{P}_\varepsilon\left(\sum_j \frac{C^2 \hat{B}_{\mu_\varepsilon^{-7/3}}\big(\xi^\varepsilon(\phi_j)\big)^{1/3}}{(1+|j|^2)^k} a^{\frac{2\gamma-4}{3}} \geq \frac{\delta'}{64}\right) \\
&\quad + \frac{1}{\alpha^2}\mathbb{P}_\varepsilon\left(\sum_j \frac{4}{(1+|j|^2)^k} \sup_{\substack{|\sigma-\tau| \leq 2\mu_\varepsilon^{-7/3} \\ \sigma, \tau \in [0, \alpha^2]}} \left|\tilde{\xi}_\tau^\varepsilon(\phi_j) - \tilde{\xi}_\sigma^\varepsilon(\phi_j)\right|^2 \geq \frac{\delta'}{64}\right),
\end{aligned}$$

with the analogous notation of (4.4) on this short time scale

$$\hat{B}_{\mu_\varepsilon^{-7/3}}(F) := \int_0^{\alpha^2} \int_0^{\alpha^2} d\sigma d\tau \, \frac{|F_\tau - F_\sigma|^b}{|\tau - \sigma|^\gamma} \mathbf{1}_{|\tau-\sigma| > \mu_\varepsilon^{-7/3}} \quad \text{with} \quad b = 6\,, \gamma = 7/3.$$

In our procedure, it was necessary to use first a time cut-off a in (4.7) in order to reduce to estimates in the kinetic time scale. Indeed the error term (4.12) occurring in the comparison with the limiting equations on the diffusive time scale $[0, T]$ was too crude to be efficient up to the smallest time scale $\mu_\varepsilon^{-7/3}$. On the kinetic scale better controls can be derived and one can show as in Lemma 6.2.6 of [10] (with the Remark 6.2.8 to take care of the large velocities) that

$$\frac{1}{\alpha^2} \mathbb{P}_\varepsilon \left(\sum_j \frac{C^2 \hat{B}_{\mu_\varepsilon^{-7/3}} \big(\tilde{\xi}^\varepsilon(\phi_j) \big)^{1/3}}{(1 + |j|^2)^k} a^{\frac{2\gamma-4}{3}} \geq \frac{\delta'}{64} \right) \leq C^2 \frac{64}{\alpha^2 \, \delta'} a^{\frac{2\gamma-4}{3}}.$$

As $a \ll \alpha$, this term vanishes in the diffusive limit. By using the proof of Lemma 6.2.5 of [10] (with the Remark 6.2.8 to take care of the logarithmic divergence), we deduce that second term vanishes also in the diffusive limit

$$\frac{1}{\alpha^2} \mathbb{P}_\varepsilon \left(\sum_j \frac{4}{(1 + |j|^2)^k} \sup_{\substack{|\sigma-\tau| \leq 2\mu_\varepsilon^{-7/3} \\ \sigma, \tau \in [0, \alpha^2]}} \big| \tilde{\xi}^\varepsilon_\tau(\phi_j) - \tilde{\xi}^\varepsilon_\sigma(\phi_j) \big|^2 \geq \frac{\delta'}{64} \right) \leq \frac{C}{\alpha^2} \mu_\varepsilon^{-1/3} \to 0.$$

Combining the previous results, (4.21) holds. This completes the proof of Proposition 4.1. □

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